Laboratorio progettuale di calcolo strutturale

Longitudinal vibrations of an elastic bar.*

Stefano Miccoli

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Consider an homogeneous elastic bar of length l, mass M, and longitudinal stiffness k; thus $\bar{N} = k\,\Delta l$ is the axial force that causes a static elongation Δl over the whole length l of the bar. The first end of the bar is fixed, while at the second end a suspended mass m is attached.

Let $u(x,t)$ be the longitudinal displacement at a point $x \in [0,l]$ along the bar. The mean unit elongation $\bar{\varepsilon}(t) = \Delta l/l$ can be expressed as

$$\bar{\varepsilon}(t) = \frac{u(l,t) - u(0,t)}{l},$$

while at point x the unit elongation $\varepsilon(x,t)$ is

$$\varepsilon(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} = \frac{\partial u(x,t)}{\partial x}. \tag{1}$$

If the unit elongation is constant along x at a time t, i. e. $\varepsilon(x,t) = \bar{\varepsilon}(t)$, we have

$$\bar{N}(t) = k\,\Delta l = kl\frac{\Delta l}{l} = kl\,\bar{\varepsilon}(t);$$

more generally, if the bar is homogeneous along x, we may assume

$$N(x,t) = kl\,\varepsilon(x,t), \tag{2}$$

where $N(x,t)$ is the axial force at point x and time t.

The equilibrium of a generic bar element with respect to translation may be written as

$$-N(x,t) + N(x+\Delta x,t) - \int_x^{x+\Delta x} \frac{M}{l}\frac{\partial^2}{\partial t^2}u(\tilde{x},t)\,d\tilde{x} = 0,$$

where $M/l = $ const. is the lineic mass. In differential form this equation becomes

$$\frac{\partial N(x,t)}{\partial x} - \frac{M}{l}\frac{\partial^2}{\partial t^2}u(x,t) = 0. \tag{3}$$

Combining eqs. (1)–(3) one obtains

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t), \tag{4}$$

with

$$\frac{1}{c^2} = \frac{M}{kl^2}.$$

Constant $c = l\sqrt{\frac{k}{M}}$ is the longitudinal wave speed: therefore

$$T_{\mathrm{w}} = \frac{l}{c} = \sqrt{\frac{M}{k}}, \tag{5}$$

is the time needed by an elastic wave to travel from one and to the other of the bar. Equation (4) is known as the *wave equation*.[1] To solve this equation for steady state vibrations one needs boundary conditions at $x = 0$ and $x = l$, while initial conditions at say $t = 0$ are not relevant.

The $x = 0$ end is fixed, so that

$$u(0,t) = 0, \qquad \forall t. \tag{6}$$

At $x = l$ there is equilibrium between the suspended mass inertial force, $-m\frac{\partial^2}{\partial t^2}u(l,t)$, and the axial force $N(l,t) = kl\,\frac{\partial u}{\partial x}(x,t)|_{x=l}$:

$$kl\frac{\partial u}{\partial x}(x,t)|_{x=l} = -m\frac{\partial^2}{\partial t^2}u(l,t).$$

Taking into account eqn. (5), this condition can be rewritten as

$$l\frac{\partial u}{\partial x}(x,t)|_{x=l} = -\frac{m}{M}\,T_{\mathrm{w}}^2\,\frac{\partial^2}{\partial t^2}u(l,t), \qquad \forall t, \tag{7}$$

which is the desired boundary condition at $x = l$.

By separation of variables one can look for a solution of the form

$$u(x,t) = X(x)\,A(t) \tag{8}$$

where $X(x)$ is a real-valued function, with the dimensions of a length, representing the time-invariant "shape" of the displacement; the real function $A(t)$ is a dimensionless, time-dependent, amplitude. Substituting the above assumption in (4), gives rise to

$$A(t)\frac{d^2}{dx^2}X(x) = \frac{1}{c^2}X(x)\frac{d^2}{dt^2}A(t),$$

[1] Please note that equations (4) and (5) are still valid when $M = 0$: one has $1/c = 0$, i. e. the r. h. s. is vanishing.

or

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x) = \frac{1}{c^2}\frac{1}{A(t)}\frac{d^2}{dt^2}A(t).$$

This latter expression can be rewritten in a dimensionless form by using eqn. (5) as

$$\frac{l^2}{X(x)}\frac{d^2}{dx^2}X(x) = \frac{T_{\mathrm{w}}^2}{A(t)}\frac{d^2}{dt^2}A(t).$$

The above equations holds for any x and t, therefore the l. h. s. and the r. h. s. have to be constant and equal:

$$\begin{cases} \dfrac{l^2}{X(x)}\dfrac{d^2}{dx^2}X(x) = p, \\ \dfrac{T_{\mathrm{w}}^2}{A(t)}\dfrac{d^2}{dt^2}A(t) = p, \end{cases}$$

where p is a dimensionless real number. By posing $p = -\alpha^2$ one has to solve

$$\begin{aligned} \frac{d^2}{dx^2}X(x) &= -\frac{\alpha^2}{l^2}X(x), \\ \frac{d^2}{dt^2}A(t) &= -\frac{\alpha^2}{T_{\mathrm{w}}^2}A(t), \end{aligned}$$

and the general solution is

$$\begin{aligned} X(x) &= U_1 \cos\left(\alpha\frac{x}{l}\right) + U_2 \sin\left(\alpha\frac{x}{l}\right), \\ A(t) &= A_1 \cos\left(\alpha\frac{t}{T_{\mathrm{w}}}\right) + A_2 \sin\left(\alpha\frac{t}{T_{\mathrm{w}}}\right). \end{aligned}$$

It is easy to show that the solution which represents steady state vibrations is attained when α and U_1, U_2, A_1, A_2 are all real, i. e. $p \leq 0$.[2]

The integration constants U_1, U_2, A_1, A_2, and parameter α have to be determined from the boundary conditions. Eqn. (6) implies $U_1 = 0$, while eqn. (7) becomes

$$\alpha\,\cos(\alpha)\,U_2 A(t) = \frac{m}{M}\alpha^2\,\sin(\alpha)\,U_2 A(t);$$

the trivial solution $\alpha = 0$, which gives rise to $u(x,t) = 0$, can be neglected. In order to solve this equation we assume $m \neq 0$ and $\alpha \neq \frac{1}{2}\pi + n\pi$, obtaining

$$\alpha\,\tan\alpha = \frac{M}{m}, \tag{9}$$

[2]When $p > 0$, α becomes imaginary; nevertheless the given solution is still valid: remembering that $\cos(i\theta) = \cosh\theta$ and $\sin(i\theta) = i\,\sinh(\theta)$ it is possible to choose integration constants in order to have real $X(x)$ and $A(t)$. However, since the hyperbolic trigonometric functions are non periodic, the case $p > 0$ is not relevant to the present problem.

which shows that α depends only on the M/m ratio. The remaining integration constants are not determined, but it is possible to rewrite solution (8) as

$$u(x,t) = U_0 \sin\Big(\alpha\frac{x}{l}\Big)\ \cos\Big(\alpha\frac{t}{T_\mathrm{w}} + \varphi\Big), \tag{10}$$

with

$$-U_0 \sin\varphi = U_2 A_2, \qquad\qquad U_0 \cos\varphi = U_2 A_1,$$

actually showing that, as one might expect, the amplitude and the phase angle of the vibration are not determined. Without loss of generality we will assume in the sequel that $\varphi = 0$.

We will first discuss the case in which $M \neq 0$, i. e. $\alpha \neq n\pi$. If $\bar{U}$ is the elongation at $x = l$, the solution (10) can be written as

$$u(x,t) = \frac{\bar{U}}{\sin\alpha}\ \sin\Big(\alpha\frac{x}{l}\Big)\ \cos\Big(\alpha\frac{t}{T_\mathrm{w}}\Big), \tag{11}$$

where α is the solution to eqn. (9), with $m \neq 0$. From eqn. (11) it is clear that α and $-\alpha$ represent the same solution: therefore only the positive solutions, $\alpha > 0$, have to be considered. For a given M/m ratio equation (9) has infinite positive solutions, confirming that a continuous system has infinite vibrational modes.

For each mode the period T of the oscillation is given by

$$T = \frac{2\pi}{\alpha}\,T_\mathrm{w}, \tag{12}$$

while

$$v(x) = \frac{1}{\sin\alpha}\ \sin\Big(\alpha\frac{x}{l}\Big) \tag{13}$$

is the modal shape.

For a vanishing suspended mass, $m = 0$, we have

$$\alpha = \frac{\pi}{2} + n\pi,$$

with $n = 0, 1, 2, \ldots$. It is easy to show that eqn. (11) is still valid and satisfies all problem equations: (4), (6), and (7) with $m = 0$. Defining T_f as the period of the fundamental mode ($n = 0$) for a vanishing suspended mass, one has

$$T_\mathrm{f} = 4T_\mathrm{w} = 4\sqrt{\frac{M}{k}};$$

the corresponding shape becomes $v(x) = \sin(\frac{\pi}{2}x/l)$.

On the contrary, when $M \to 0$, $\alpha \to n\pi$. To compute the limiting value of the period, since $T_\mathrm{w} \to 0$, eqn. (12) has to be rewritten as

$$T = \frac{2\pi}{\alpha}\sqrt{\frac{m}{k}\frac{M}{m}} = 2\pi\sqrt{\frac{m}{k}}\ \sqrt{\frac{\tan\alpha}{\alpha}};$$

taking the limit one has

$$\lim_{\alpha \to n\pi} \sqrt{\frac{\tan\alpha}{\alpha}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

showing that

$$T_{\mathrm{o}} = 2\pi\sqrt{\frac{m}{k}}, \tag{14}$$

is the period of the fundamental mode ($n = 0$) when the mass M of the bar vanishes. The limiting shape of this mode is $v(x) = x/l$; again all equations are satisfied, showing that this is a legitimate solution. On the contrary, higher order modes degenerate to the trivial $u(x,t) = 0$ solution; in other terms the solution of the wave equation for the massless elastic bar reduces to the solution of the simple harmonic oscillator.

It can be convenient to express the period T of the fundamental mode in a form which is similar to the period of the harmonic oscillator:

$$T = 2\pi\sqrt{\frac{m + M_{\mathrm{eq}}}{k}}, \tag{15}$$

where M_{eq} is an *equivalent* mass, which accounts for the elastic bar mass. By equating (12) and (15),

$$\frac{2\pi}{\alpha}\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{m + M_{\mathrm{eq}}}{k}},$$

one has, exploiting eqn (9),

$$M_{\mathrm{eq}} = M\left(\frac{1}{\alpha^2} - \frac{m}{M}\right) = M\left(\frac{1}{\alpha^2} - \frac{1}{\alpha\tan\alpha}\right).$$

Defining a coefficient β,

$$\beta = \frac{1}{\alpha^2} - \frac{1}{\alpha\tan\alpha}, \tag{16}$$

which is via eqn. (9) a function of m/M, one finally has

$$T = 2\pi\sqrt{\frac{m + \beta M}{k}}. \tag{17}$$

Values of β as a function of m/M are given in figure 1.

It is a simple calculus exercise to show that

$$\lim_{\alpha \to 0} \beta = \frac{1}{3}, \qquad \lim_{\alpha \to \frac{\pi}{2}} \beta = \frac{4}{\pi^2};$$

more specifically

$$0{,}405 \approx \frac{4}{\pi^2} > \beta > \frac{1}{3} \approx 0{,}333$$

where bigger β corresponds to $M \gg m$, and smaller β to $M \ll m$.

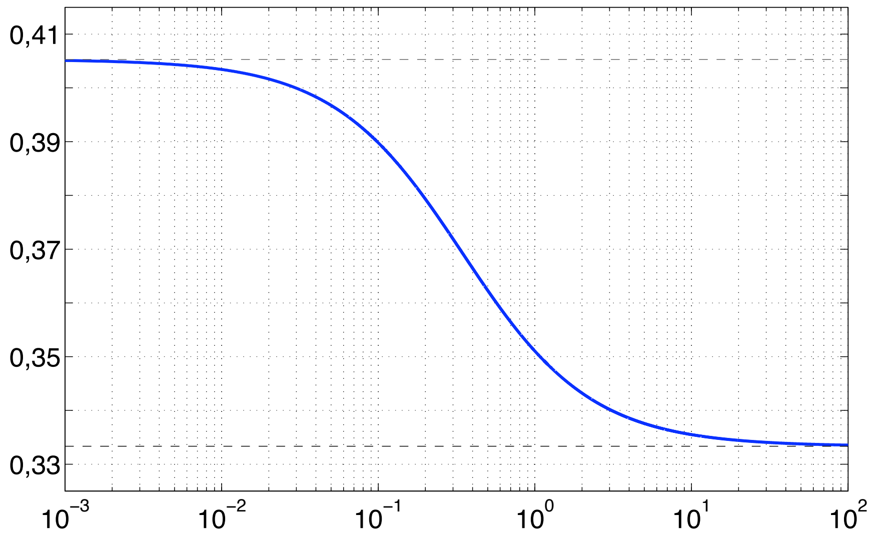

Figure 1: Values of β as a function of m/M.

It should be clear that period T given by eqn. (17) is exact, if the correct value of β, defined by eqn. (9) and (16) and represented in fig. 1, is used. In practice, when $m \gg M$, a common first order approximation is to assume $\beta = \frac{1}{3}$, or

$$T_{\mathrm{o},1} = 2\pi\sqrt{\frac{m + \frac{1}{3}M}{k}}. \tag{18}$$

Thus when computing the period of oscillation of a suspended mass, one can use

- the exact expression T, given by eqn. (12) or eqn. (17),
- the approximate value $T_{\mathrm{o},1}$ of eqn. (18), valid for $m \gg M$, or
- neglect altogether mass M and use T_{o}, eqn. (14).

These last two assumptions are associated with relative errors $|T_{\mathrm{o}} - T|/T$ and $|T_{\mathrm{o},1} - T|/T$ which are depicted in figure 2.

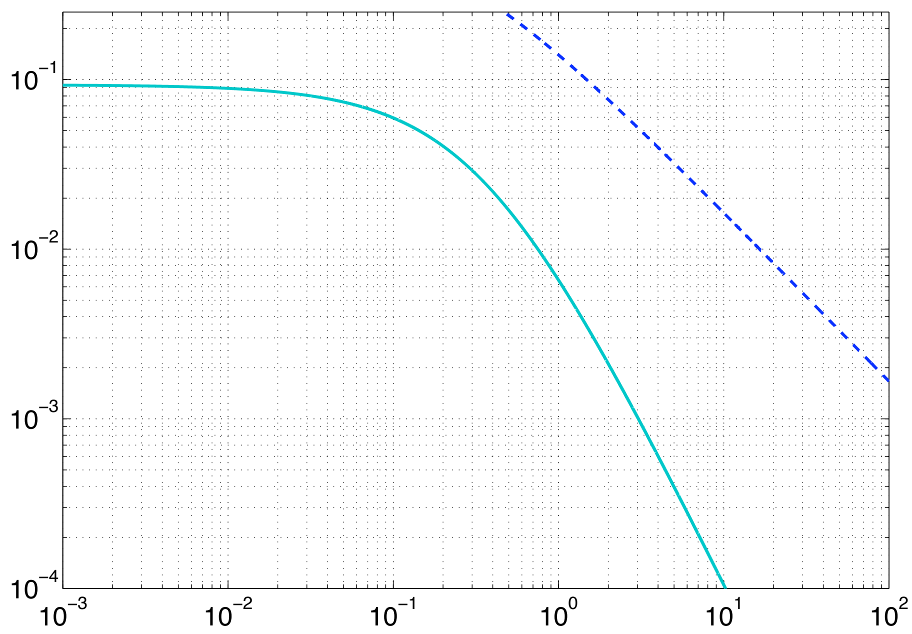

Figure 2: Errors $|T_{\mathrm{o},1} - T|/T$ (solid line) and $|T_{\mathrm{o}} - T|/T$ (dashed line) as a function of m/M.